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Induced magnetic monopole from trapped Λ-type atom

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Abstract. We investigate the spatial motion of the trapped atom with the electromagnetically induced transparency (EIT) configuration where the two Rabi transitions are coupled to two classical light fields respectively with the same detuning. When the internal degrees of freedom can be decoupled adiabatically from the spatial motion of the center of mass via the Born-Oppenheimer approximation, it is demonstrated that the lights of certain profile can provide the atom with an effective field of magnetic monopole, which is the so-called induced gauge field relevant to the Berry's phase. Such an artificial magnetic monopole structure manifests itself in the characterizing energy spectrum.

PACS. 03.65.Vf Phases: geometric; dynamic or topological – 42.50.Vk Mechanical effects of light on atoms, molecules, electrons, and ions – 14.80.Hv Magnetic monopoles

Modern concept of magnetic monopole in quantum mechanics was postulated by Dirac in 1931 [1]. Since that time physicist have been making efforts to seek for the magnetic monopole in real space for more than seventy years. Though convincing evidence for its existence in real space has not yet been found, the theoretical conception of magnetic monopole has initiated many important progresses in both physics and mathematics. In fact, in the extremely high energy scale that we have not reached at present, the grand unification theory [2] predicted the magnetic monopole as a consequence of the beautiful topology structure of Yang-Mills theory [3]. The discovery of Berry phase also resulted in a physical implementation of magnetic monopole in the parameter space [4]. Precisely speaking, the degeneracy point in the parameter space acts like a magnetic monopole caused by an effective gauge field. The Berry phase based magnetic monopole of this kind can be demonstrated in association with the anomalous Hall effect of ferromagnetic metals [5] where the slowly changing parameter is just the crystal momentum. Now we consider an artificial realization of magnetic monopole in real space.

For the quantum adiabatic process induced Berry phase, it is well-known that when the slowly varying parameters are the dynamic variables of a subsystem interacting with another subsystem with fast varying variables, the adiabatic separation of the two subsystems via the Born-Oppenheimer approximation [6] can provide the slow subspace with a scaler and a vector potentials [7] called the induced gauge potential. In the neutron spin precession experiment [8] the Aharonov-Borhm effect caused by this vector potential was pointed out as a manifestation of Berry phase. It is also recognized that if we have a quantum system which can be considered as a bead sliding along a rod which rotates slowly, the fast motion of the bead can induce into the dynamics of the bead a gauge potential which has the same form of the one of a magnetic monopole [9]. One example is the induced gauge field of nuclear rotation in a diatom [10]. Up to our knowledge this is the unique example of the physical implementation of magnetic monopole in the real space. It is pointed out that in this system the parameters of the "magnetic monopole" is determined by the character of the molecular system and then can not be adjusted.

In this article, we will derive the monopole type induced gauge field for the spatial motion of $\Lambda-$type atom interacting with control and probe laser beams which drive the transitions $|e\rangle$-$|1\rangle$ and$|e\rangle$-$|2\rangle$ respectively (see Fig. 1) [11,12]. When the atom is cold enough, its internal degrees of freedom can decouple adiabatically from the spatial motion of the center of mass. Correspondingly, the Born-Oppenheimer approximation provides the atomic center of mass with an effective gauge field. This effect have been studied by some authors [13–15]. Here we will show that when the lights are artificially shaped in certain profiles, effective magnetic monopole field can be created as a special induced gauge field relevant to the Berry phase. We predict that such an artificial magnetic monopole can be observed experimentally through its special spectral structure. In fact, the prompt advance in experiments of trapping and cooling atoms has provided a platform to test our predictions exactly.

The Hamiltonian for our cold atom system driven by two laser beams (see Fig. 1) can be written in the form

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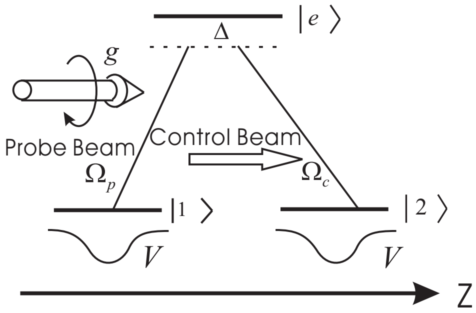

Fig. 1. Three level atoms interacting with two laser beams. The probe beam coupling the states $|e\rangle$ and $|1\rangle$ has an orbital angular momentum. The atoms in state $|1\rangle$ and $|2\rangle$ are trapped by the potential V, which is a function of the atomic position $\mathbf{r}$.

$H = \mathbf{P}^2/2M + H_f(\mathbf{r})$ with the local internal Hamiltonian

$$H_f(\mathbf{r}) = \sum_{i=1}^{2} V_i(\mathbf{r})\,|i\rangle\langle i| + \Delta\,|e\rangle\langle e| + \Omega_p(\mathbf{r})\,|e\rangle\langle 1| + \Omega_c(\mathbf{r})\,|e\rangle\langle 2| + h.c. \quad (1)$$

Here, $\mathbf{r}$ is the atomic position, Δ the one-photon detunning, $\Omega_c(\mathbf{r})$ ($\Omega_p(\mathbf{r})$) the Rabi frequency of the probe (control) beam, and $V_i(\mathbf{r})$ the trap potential of the ith inner energy level. We assume that there is no trap for the level e, i.e., $V_e(\mathbf{r}) = 0$, and there is the same trap for both level 1 and 2, i.e., $V_1(\mathbf{r}) = V_2(\mathbf{r}) = V(\mathbf{r})$. This assumption just ensures the occupancy of the dark state.

To use the generalized Born-Oppenheimer approximation [8], we first diagonalize the interaction part $H_f(\mathbf{r})$ of the Hamiltonian and obtain the $\mathbf{r}$-dependent eigenvalues of $H_f(\mathbf{r})$: $E_0(\mathbf{r}) = V(\mathbf{r})$ and

$$E_\pm(\mathbf{r}) = \frac{1}{2}(\tilde{\Delta} \pm \sqrt{4|\Omega_c|^2 + 4|\Omega_p|^2 + \tilde{\Delta}^2}) + V(\mathbf{r}). \quad (2)$$

where $\tilde{\Delta} = \tilde{\Delta}(\mathbf{r}) = \Delta - V(\mathbf{r})$ is the local one-photon detunning. The eigenstate corresponding to $E_0(\mathbf{r})$ is the $\mathbf{r}$-dependent dark sate defined as

$$|D(\mathbf{r})\rangle = \frac{1}{\Omega}\left[\Omega_p|2\rangle - \Omega_c|1\rangle\right]. \quad (3)$$

where $\Omega = \Omega(\mathbf{r}) = \sqrt{|\Omega_c|^2 + |\Omega_p|^2}$. The other two eigenstates corresponding to the eigenenergies $E_\pm(\mathbf{r})$ can be noted as $|B_\pm(\mathbf{r})\rangle$. Their explicit expressions are not necessary to the following discussion for the case that the atom spatial motion is sufficiently slow so that the internal motion is not excited.

It is well-known that the atomic wave function in the $\mathbf{r}$-representation can be written as

$$\langle\mathbf{r}|\Psi\rangle = \psi_0(\mathbf{r})\,|D(\mathbf{r})\rangle + \sum_{k=+,-}\psi_k(\mathbf{r})\,|B_k(\mathbf{r})\rangle. \quad (4)$$

When the energy gaps $E_\pm - E_0(\mathbf{r})$ between the dark state $|D(\mathbf{r})\rangle$ and the states $|B_\pm(\mathbf{r})\rangle$ are large enough, the Born-Oppenheimer approximation is applicable. Under this approximation, the atom can be assumed to be "kept" in the dark state $|D(\mathbf{r})\rangle$ at every position $\mathbf{r}$ and the eigen wave function of the total Hamiltonian H can be written as $\langle\mathbf{r}|\Psi\rangle = \psi_0(\mathbf{r})\,|D(\mathbf{r})\rangle$ where $\psi_0(\mathbf{r})$ satisfies the eigenequation

$$\left[\frac{1}{2M}(\mathbf{P} + \mathbf{A}_0)^2 + V(\mathbf{r})\right]\psi_0(\mathbf{r}) = E\psi_0(\mathbf{r}). \quad (5)$$

Here, $\mathbf{A}_0(\mathbf{r}) = -i\langle D(\mathbf{r})|\nabla|D(\mathbf{r})\rangle$ is the induced gauge potential corresponding to the dark state.

As in reference [15], we may assume that the probe beam and the control beam have the same frequency and propagate along the z-direction. Then the Rabi frequencies $\Omega_p(\mathbf{r})$ and $\Omega_c(\mathbf{r})$ can be expressed as

$$\Omega_p = |\Omega_p|\exp\left[i(\mathbf{k}\cdot\mathbf{r} + g\phi)\right], \quad (6)$$
$$\Omega_c = |\Omega_c|\exp(i\mathbf{k}\cdot\mathbf{r}).$$

Here, the real parameters $|\Omega_p(\mathbf{r})|$ and $|\Omega_c(\mathbf{r})|$ are just the slowly varying norms of the Ω_p and Ω_c and ϕ the directional angle of the $(x-y)$-plane. Note that in writing down equation (6), we have also assumed that the probe beam has an orbital angular momentum $g\phi$ with g an integer [16]. Then the induced gauge potential $\mathbf{A}_0$ of the dark state can be written as

$$\mathbf{A}_0 = \frac{g|\Omega_p|^2}{\Omega^2}\nabla\phi. \quad (7)$$

It is obvious that the orbital angular momentum number g of the probe beam induces an effective magnetic monopole of strength g. We thus conclude that the effective magnetic charge of this induced monopole can be controlled artificially by adjusting the angular momentum of photons in the probe beam.

When the norms $|\Omega_p(\mathbf{r})|$ and $|\Omega_c(\mathbf{r})|$ of the Rabi frequencies take the forms

$$|\Omega_p|^2 = \xi(r+z), \quad |\Omega_c|^2 = \xi(r-z) \quad (8)$$

where ξ is a constant positive coefficient [17], the potential $\mathbf{A}_0$ has the same form as that of the potential created by a monopole and can be expressed as

$$\mathbf{A}_0 = \frac{g(1+\cos\theta)}{2r\sin\theta}\mathbf{e}_\phi. \quad (9)$$

Here, r, θ and ϕ are just the spherical polar coordinates. In this case we have $|\Omega_p| = |\Omega_c| = 0$ at the origin $\mathbf{r} = 0$. This leads to an "accidental degeneracy" $E_\pm(\mathbf{0}) = E_0(\mathbf{0})$ at the origin where the energy levels cross. Thus the Born-Oppenheimer approximation does not work well. For this reason, in the following discussion, we only discuss the atomic motion around the region far away from the origin.

For the original Dirac's monopole, as is well-known, the induced monopole potential $\mathbf{A}_0$ has singularity at the string of $\theta = 0$. But this singularity can be exorcized by means of the approach developed by Wu and Yang [3,19]. To this end one needs to divide the total real space (except the origin) into two regions (see Fig. 2), R_a: $0 \le \theta < \pi/2+\delta$ and R_b: $\pi/2-\delta \le \theta < \pi$, which are two overlapping

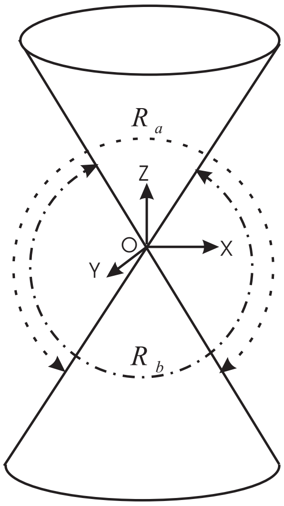

Fig. 2. The whole space excluding the origin O is divided into two regions R_a and R_b. R_a is the space excluding the lower circular cone (the region enveloped by the dashed line), R_b is the space excluding the upper one (the region enveloped by the dash dot line). The space outside of the two cones is the overlap of R_a and R_b.

caps and have their well defined local coordinates. Then the dark state $|D(\mathbf{r})\rangle$ and the wave function $\psi_0(\mathbf{r})$ can no longer be defined globally. Instead, $|D(\mathbf{r})\rangle$ and $\psi_0(\mathbf{r})$ will have different expressions in the different regions marked by a and b. For instance the dark state can be defined as

$$|D(x)\rangle_a = e^{-i(g\phi)} \frac{1}{\Omega} (\Omega_p |2\rangle - \Omega_c |1\rangle) \tag{10}$$

in R_a and

$$|D(x)\rangle_b = \frac{1}{\Omega} (\Omega_p |2\rangle - \Omega_c |1\rangle) \tag{11}$$

in R_b. The Schrödinger equation (5) can be rewritten in different caps as

$$\left[\frac{1}{2M} (\mathbf{P} + \mathbf{A}_{0\alpha})^2 + V(\mathbf{r})\right] \psi_{0\alpha}(\mathbf{r}) = E\psi_{0\alpha}(\mathbf{r}) \tag{12}$$

in R_α for $\alpha = a, b$. Here, $\psi_{0a}(\mathbf{r})$ and $\psi_{0b}(\mathbf{r})$ are the expressions of $\psi_0(\mathbf{r})$ in R_a and R_b respectively and the gauge potential $\mathbf{A}_{0a}$ ($\mathbf{A}_{0b}$) can be expressed as

$$\mathbf{A}_{0a} = \frac{g(-1+\cos\theta)}{2r\sin\theta} \hat{e}_\phi, \quad \mathbf{A}_{0b} = \frac{g(1+\cos\theta)}{2r\sin\theta} \hat{e}_\phi. \tag{13}$$

Apparently, $\mathbf{A}_{0a}$ ($\mathbf{A}_{0b}$) is not singular in the region R_a (R_b). In the overlap of R_a and R_b, we have a connection $\psi_{0b}(\mathbf{r}) = \psi_{0a}(\mathbf{r}) e^{-ig\phi}$ due to the $U(1) - guage$. Such two local wave functions in two caps with this connection in the overlapping region are mathematically called the wave section.

We now consider a simple case that the trap potential $V(\mathbf{r})$ is spherically symmetrical. In this case we can separate the radial degree of freedom r and the angular degrees

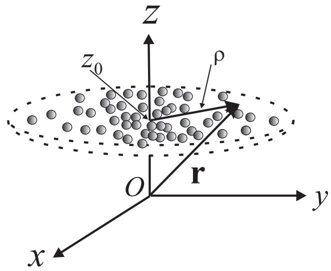

Fig. 3. The atoms are trapped in the "discus form" region whose center is the point $(0, 0, z_0)$. In this figure, O is the origin of the coordinate.

of freedom θ and ϕ in the Schrödinger equation (12) by invoking the generalized angular momentum operator [19]

$$\mathbf{L} = \mathbf{r} \times (\mathbf{P} + \mathbf{A}_0) - \frac{g\mathbf{r}}{2r}. \tag{14}$$

It was proved [19] that $\psi_0(\mathbf{r}) = R_l(r) Y_{q,l,m}$ can be expressed in terms of the monopole harmonics $Y_{\frac{g}{2},l,m}(\theta, \phi)$ which is the common "*eigensection*" of L^2 and L_z with respect to the eigenvalues $l(l+1)$ and m. Here, we have $l = |g/2|, |g/2| + 1, ...$ and $m = -l, -l+1, ..., l$. The radial wave function $R_l(r)$ satisfies the equation

$$\left[-\frac{\partial_r (r^2 \partial_r)}{2Mr^2} + \frac{l(l+1) - (g/2)^2}{2Mr^2} + V(r) - E\right] R_l = 0. \tag{15}$$

In the special case $V = 0$, when $E > 0$, the solution of equation (15) is [20] a Bessel function $R = (1/\sqrt{kr}) J_\mu(kr)$ with

$$\mu = \sqrt{l(l+1) - \left(\frac{g}{2}\right)^2 + \frac{1}{4}}, \quad k = \sqrt{2ME}. \tag{16}$$

Next we consider a more interesting case. We assume the the trap is a harmonic potential

$$V = \frac{1}{2} M\omega_z^2 (z - z_0)^2 + \frac{1}{2} M\omega^2 \rho^2 \tag{17}$$

where $z_0 > 0$ and $\rho^2 = x^2 + y^2$ (see Fig. 3). In this case, the atom is confined near a fixed point $(0, 0, z_0)$ in the region R_a. Therefore, we need only consider the expression ψ_{0a} of ψ_0 in R_a. Since the trap potential is cylindrically symmetrical, it is convenient to discuss this problem with the cylindrical coordinate (ρ, z, ϕ). Because of the cylindrical symmetry of V and $\mathbf{A}_0$, the wave function ψ_{0a} has the factor $\exp(im\phi)$ $(m = 0, \pm1, \pm2, ...)$ which is just the generator of the rotation along z-axis. Then we have $\psi_{0a} = T_m(\rho, z) \exp(im\phi)$ where $T_m(\rho, z)$ satisfies the radial Schrödinger equation

$$\frac{1}{2M} \left[-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \partial_z^2 + F_m(\rho, z)\right] T_m + VT_m = ET_m. \tag{18}$$

Here, the function $F_m(\rho, z)$ is defined as

$$F_m(\rho, z) = \left(\frac{m}{\rho} + g\frac{z - \sqrt{\rho^2 + z^2}}{2\sqrt{\rho^2 + z^2}\rho}\right)^2. \quad (19)$$

If the trap along z-axis is strong enough, i.e., ω_z is large enough, we can make the approximation $F_m(\rho, z) \approx F_m(\rho, z_0)$. It is obvious that $F_m(\rho, z_0)$ can be expanded as a Laurent series of ρ. We also assume that the trap localized in $(x - y)$-plane is also strong enough that we can only keep $F_m(\rho, z_0)$ up to the term proportional to ρ^2. Thus approximately we have

$$F_m(\rho, z) \approx \frac{m^2}{\rho^2} + \frac{g^2}{16z_0^4}\rho^2 - \frac{mg}{2z_0^2}, \quad (20)$$

and we can solve equation (18) to obtain the energy spectrum

$$E_{m,n_\rho,n_z} = (2n_\rho + |m| + 1)\,\widetilde{\omega} - \frac{mg}{4Mz_0^2} + \left(n_z + \frac{1}{2}\right)\omega_z \quad (21)$$

in terms of the radial quantum number n_ρ and the vertical one n_z $(= 0, 1, 2, ...)$, where

$$\widetilde{\omega} = \sqrt{\omega^2 + \frac{g^2}{16M^2z_0^4}} \quad (22)$$

is the modified radial frequency for the two dimensional reduced radial oscillator. The corresponding radial frequency shift $\widetilde{\omega} - \omega \sim g^2/32\omega M^2 z_0^4$ can be regarded as the first observable effect of the artificial magnetic monopole. The additional term $-mg/4Mz_0^2$ in the energy spectrum of the spatial motion of atom may reflect its effect in realistic experiment. The corresponding wave function can also be obtained explicitly:

$$\psi_{0a}^{m,n_\rho,n_z} = N_{n_z} e^{im\phi}\rho^{|m|} e^{-M(\widetilde{\omega}+\omega_z)\rho^2/2} \times F\left(-n_\rho, |m| + 1, M\widetilde{\omega}\rho^2\right) H_{n_z}\left(\sqrt{M\omega_z}z\right). \quad (23)$$

Here, $N_{n_z} = \left[\sqrt{M\omega_z}/\sqrt{\pi}2^{n_z}n_z!\right]^{\frac{1}{2}}$, F is the confluent hypergeometric function and H_{n_z} the Hermit function.

Since the above results are achieved with the generalized Born-Oppenheimer approximation, we should investigate the condition under which this approximation is applicable. The adiabatic condition can be obtained semiclassically. For simplicity, we only consider the case $\Delta = 0$. In our problem, the sufficient condition of adiabatic approximation is

$$|\langle D|\nabla|B_\pm\rangle \cdot \mathbf{v}| \cdot |E_\pm - E_0|^{-1} \ll 1, \quad (24)$$

where $\mathbf{v}$ is the velocity of the atomic center of mass (c.m). By straightforward calculation, it can be obtained that the upper limit of $|\langle D|\nabla|B_\pm\rangle \cdot \mathbf{v}|$ is $(1/2r)(|v_\rho| + |v_z| + g|v_\phi|) \sim gv/r$. Here, v_ρ, v_z and v_ϕ are the components of the c.m velocity in the cylindrical coordinate and v the speed rate. On the other hand, the lower limit of $|E_\pm - E_0|$ is $|\sqrt{|\Omega_c|^2 + |\Omega_p|^2 + (V/2)^2} - V/2|$. Then the condition (24) can be rewritten as

$$\frac{g}{r}\sqrt{\frac{2E}{M}} \ll \frac{1}{2}(\sqrt{8\xi r + E^2} - E). \quad (25)$$

Here we have used the fact that the upper limit of v is $\sqrt{2E/M}$ and the upper limit of V is the atomic energy E.

The adiabatic condition (25) can be satisfied in some realistic cases. For instance, we confider the cesium atoms trapped around the origin $r = 0$. We assume $\xi = \pi \times 10^{17}$ (Hz)^{2}m^{-1} and the atomic energy $E \sim 10^{-27}$ J. Then it is easy to see that when the optical angular momentum $g \sim 10^1$, the adiabatic condition is satisfied when $r > 10^{-4}$ m. Therefore, if the scale of the atomic ensemble is $r \sim 10^{-3}$ m, the adiabatic approximation holds in almost all the region of the atomic motion. Another case discussed above is that the cesium atoms are trapped around the point $(0, 0, z_0)$ by the harmonic potentials. We assume $z_0 \sim 10^{-4}$ m, $g \sim 10^2$ and the frequencies of the trap potentials are $\omega_z \sim 10^6$ Hz and $\omega \sim 10^2$ Hz. Then the adiabatic condition can be met again if E and ξ has the same values as mentioned above. Therefore, the induced change of zero point energy $-mg/\left(4Mz_0^2\right)$ is about 10^1 Hz. The frequency shift $\widetilde{\omega} - \omega$ caused by the monopole potential will have the same order with ω.

Before concluding this paper we make a remark on the results that we have obtained. We assume a group of more universal conditions $|\Omega_p|^2 = \xi(r + z)$, $|\Omega_c|^2 = \xi[(2\eta - 1)r - z]$ where $\eta > 1$ are satisfied. In this case, the induced gauge potential will be $\mathbf{A}_0 = g(1 + \cos\theta)\,\hat{e}_\phi/(2\eta r\sin\theta)$. As we have shown above, to eliminate the singularity of $\mathbf{A}_0$, a relationship between the dark state and wave function in the regions R_a and (R_b) should be introduced: $|D\rangle_a = e^{-ig\phi}|D\rangle_b$, $\psi_{0a}(\mathbf{r}) = \psi_{0b}(\mathbf{r})\,e^{ig\phi/\eta}$. It is apparently that ifg/η is not an integer, the dark state and wave function are not single valued function in R_a (but their product is single valued). Therefore, when solving the Schrödinger equation (12) in R_a we should change the single valued condition $\psi_{0a}|_{\phi=0} = \psi_{0a}|_{\phi=2\pi}$ into $\psi_{0a}|_{\phi=0} = \psi_{0a}|_{\phi=2\pi}e^{i2g\pi/\eta}$. Obviously, the condition that the wave function are single valued in the whole space (g/η is an integer) is just corresponding to Dirac's quantization condition.

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17. It is pointed out that $\xi\,(r \pm z)$ is not an analytical function at the origin $\mathbf{r} = 0$. In fact we can replace r in (8) with another function $r' = (x^2 + y^2 + z^2 + \delta^2)^{1/2}$ where δ can be any real number. In this case the Rabi frequency $|\Omega_p|$ $(|\Omega_c|)$ is proportional to $[\xi\,(r' \pm z)]^{1/2}$ which is analytical in the whole spaces and then may be expanded with Laguerre-Gausse beams [18] in the region near z axes. The Born-Oppenheimer approximation is applicable in the region $r \gg \delta$ where $|\Omega_p|$ $(|\Omega_c|)$ (and then the energy spacings) is large enough. In this region, we have $r' \approx r$ and the effective monopole potential (9) is applicable
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